B.Sc. Ist Year (Hons)
(B.Sc. IInd Year Subs.)
Differential Calculus
Paper- IInd
Lecture -1 :-

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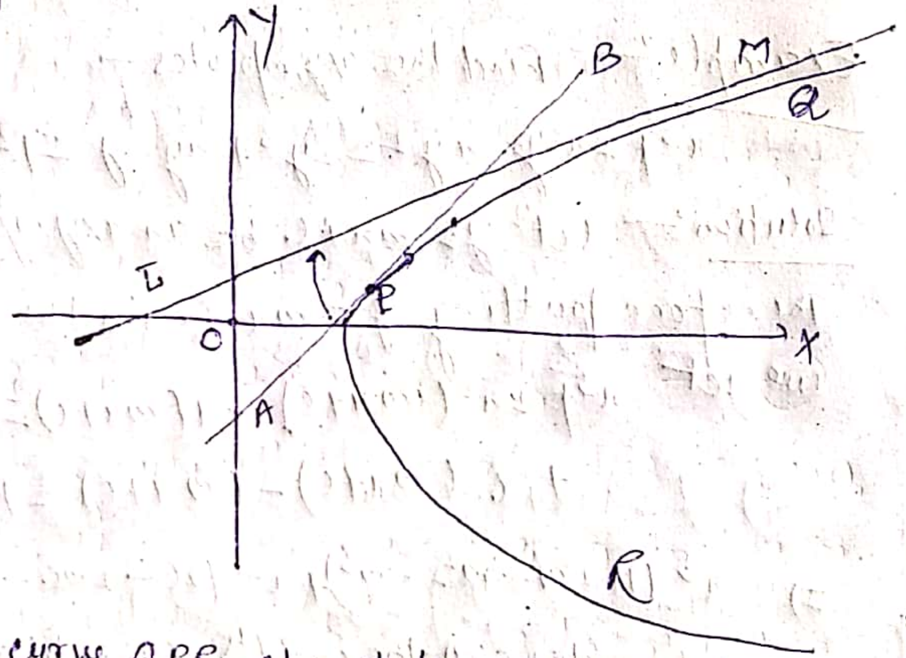
Asymptotes :-

A straight line at a finite distance from the origin to which a tangent to a curve tends, as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve.



In figure, curve QPR, a parabola and at a point P, APB is a tangent; as Q tends to infinity along the curve; tangent APB tends to LM, in this case LM is called asymptote.

Now, we study different kinds of asymptotes :-

Rectangular asymptotes :- If an asymptote to any curve is either parallel to X-axis or parallel to Y-axis, then it is called a rectangular asymptote. An asymptote parallel to x-axis is called a horizontal asymptote and an asymptote parallel to Y-axis is called a vertical asymptote.

An asymptote, which is neither parallel to X-axis nor parallel to Y-axis is called an oblique asymptote.

Conditions that the straight line $y = mx + c$ may be an asymptote to the curve $y = f(x)$:-

The eqn of tangent to the curve $y = f(x)$ at a point $P(x, y)$ is given by $Y - y = \left(\frac{dy}{dx}\right)(X - x)$

$$\Rightarrow Y = \left(\frac{dy}{dx}\right)X + \left(y - x\frac{dy}{dx}\right) \quad \text{---} (1)$$

The eqn (1) will become as an asymptote (excluding for the asymptote parallel to the y-axis) to the curve if both $\frac{dy}{dx}$ tends to a finite limit, say m and $y - x\frac{dy}{dx}$ tends to a finite limit, say c; as $x \to \infty$

$\Rightarrow y = mx + c$ would be an asymptote if

$$\lim_{x\to\infty}\left(\frac{dy}{dx}\right) = m \quad \text{and} \quad \lim_{x\to\infty}\left(y - x\frac{dy}{dx}\right) = c.$$

We can write.

$$\lim_{x\to\infty} \frac{y - x\frac{dy}{dx}}{x} = \lim_{x\to\infty} \frac{c}{x} = 0$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{y}{x}\right) - m = 0 \quad \text{as} \lim_{x\to\infty}\left(\frac{dy}{dx}\right) = m$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{y}{x}\right) = m.$$

Thus if $y = mx + c$ is an asymptote to a curve, then $m = \lim_{x\to\infty}\left(\frac{y}{x}\right)$ and $c = \lim_{x\to\infty}(y - mx)$.

Conditions for infinite Roots of an equation:-

Consider the equation $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$ — ①

be an eqn of n^{th} degree.

It is obvious that one root of the equation ① will be zero if $a_n = 0$. Again if $a_n = 0$ and $a_{n-1} = 0$ then eqn ① will have two roots as zero.

Putting $x = \frac{1}{z}$ in ①; we get

$$a_0\left(\frac{1}{z}\right)^n + a_1\left(\frac{1}{z}\right)^{n-1} + a_2\left(\frac{1}{z}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{1}{z}\right) + a_n = 0$$

$$\Rightarrow a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

$$\Rightarrow a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0 \quad \text{— ⑪}$$

If $a_0 = 0$ then eqn ⑪ will be, one value of z be zero, two roots of eqn will be zero if $a_0 = 0$, $a_1 = 0$ and so on. Since $x = \frac{1}{z}$, therefore when $z = 0$ then $x \to \infty$.

Hence one root of eqn ① will be infinite if $a_0 = 0$, two roots will be infinite if $a_0 = 0$ and $a_1 = 0$ and so on. Thus an eqn of nth degree has one root infinite if $a_0 = 0$ i.e. the coeffts of x^n is zero. It has two roots infinite if the coeffts of x^n as well as of x^{n-1} are zero and so on.

Oblique Asymptote of Algebraic curves:-

Let the eqn of a rational algebraic curve be $f(x,y) = 0$ — ①

Let $y = mx + c$ be an asymptote of the curve ①, then putting $y = mx + c$ in ①, we get an eqn of the form

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0 \quad \text{— ⑪}$$

where $a_0, a_1, \ldots, a_n$ are functions of m & c.

Solving eqn (ii), we shall find out the intersection of (i) with $y = mx + c$. But $y = mx + c$ is an asymptote and so it must intersect (i) in two points at infinity. Hence (ii), must have two infinite roots, for which we must have $a_0 = 0$ and $a_1 = 0$. We can find out the values of m & c from these two equations and in this case, we can determine the asymptote of the curve (i).

Example: Find the asymptotes of

$$x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0.$$

Solution: Let $y = mx + c$ be an asymptote of the curve. Therefore putting $y = mx + c$ in the eqn of curve, we get

$$x^3 + 2x^2(mx+c) - x(mx+c)^2 - 2(mx+c)^3 + x(mx+c) - (mx+c)^2 - 1 = 0$$

$$\Rightarrow x^3(1 + 2m - m^2 - 2m^3) + x^2(2c - 2mc - 6m^2c + m - m^2) + \dots = 0$$

Equating to zero the coeffts of two highest degree terms in x, we have

$$1 + 2m - m^2 - 2m^3 = 0 \quad \text{--- (i)}$$

and

$$2c - 2mc - 6m^2c + m - m^2 = 0 \quad \text{--- (ii)}$$

from (i); we have $m = 1, -1, -\frac{1}{2}$

from (ii); $c = 0, -1, \frac{1}{2}$ respectively.

Putting these values of m and c in $y = mx + c$, we get the equation of asymptotes are

$$y = x, \quad y = -x - 1; \quad y = -\frac{1}{2}x + \frac{1}{2}$$

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B.Sc. Ist
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Lecture - 2 :-

Asymptote :-

General working Rule for finding asymptotes :-

Step I :- Put $y = mx + c$ in the equation of curve and arrange this equation in descending powers of x.

Step II :- Equate to zero the coefficients of two highest degree terms.

Step III :- Solve first equation to get different values of m. Put these values of m in IInd equation and find the corresponding values of c.

Step IV :- Put the value of m and corresponding values of c in the equation $y = mx + c$ to get the equation of asymptotes.

Shorter method of finding the asymptotes :-

Let the equation of the curve be

$$(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + \cdots + b_ny^{n-1}) + (c_2x^{n-2} + c_3x^{n-3}y + \cdots + c_ny^{n-2}) + \cdots = 0 \quad \text{--- (I)}$$

$$\Rightarrow x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \cdots = 0$$

where $\phi_n(y/x)$ represents an expression of nth degree in y/x.

Let $y = mx + c$ ——— (II)

be an asymptote of the curve.

From eqn (II), we have

$$\frac{y}{x} = m + \frac{c}{x} \quad \text{——— (III)}$$

Using eqn (III) in eqn (I), we obtained -

$$x^n \phi_n(m+c/x) + x^{n-1}\phi_{n-1}(m+c/x) + x^{n-2}\phi_{n-2}(m+c/x) + \dots = 0 \quad \text{(iv)}$$

By Taylor's theorem, we know that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \dots$$

Expanding each term of eqn (iv) by Taylor's theorem, we get

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{c^2}{x^2}\cdot\frac{1}{\lfloor 2}\phi_n''(m) + \dots\right\}$$

$$+ x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi_{n-1}'(m) + \frac{c^2}{x^2}\cdot\frac{1}{\lfloor 2}\phi_{n-1}''(m) + \dots\right\}$$

$$+ x^{n-2}\left\{\phi_{n-2}(m) + \frac{c}{x}\phi_{n-2}'(m) + \dots\right\} = 0$$

Arranging the terms in descending powers of x, we get

$$x^n\phi_n(m) + x^{n-1}\left\{c\phi_n'(m) + \phi_{n-1}(m)\right\} + x^{n-2}\left\{\frac{c^2}{\lfloor 2}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m)\right\} + \dots = 0 \quad \text{(v)}$$

The above eqn gives the abscissae of n points of intersection of the curves and the line (ii). Since the line $y = mx + c$ is an asymptote therefore, the coefficient of first two highest degree terms will be zero i.e.

$$\phi_n(m) = 0 \quad \text{(VI)}$$

and $$c\,\phi_n'(m) + \phi_{n-1}(m) = 0 \quad \text{(VII)}$$

Equation (VI) being the nth degree in m, gives n values of m. For each value of m the corresponding value of c can be obtained from eqn (VII), which gives

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)} \quad \text{(VIII)}$$

Thus we get the an asymptote of the curve for every pair of m and c of the form $y = mx + c$.

Example: Find all the asymptotes of the curves

$$x^2+2xy-y^2+x-y+2=0$$

Solution: Putting $x=1$, $y=m$ in the highest (second) degree terms of the given equation of the curve; then

$$\phi_2(m)=1+2m-m^2$$

Also $\phi_2'(m)=2-2m$

To determine values of m; we put $\phi_2(m)=0$

$$\Rightarrow m^2-2m-1=0$$

$$\Rightarrow m=\frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}$$

Again; putting $x=1$; $y=m$ in the first degree terms of the equation of the given curves; we get

$$\phi_1(m)=1-m$$

Now; $$c=-\frac{\phi_{n-1}(m)}{\phi_n'(m)}=-\frac{\phi_1(m)}{\phi_2'(m)}$$

$$\Rightarrow c=-\tfrac{1}{2}$$

Hence the asymptotes are $y=(1\pm\sqrt{2})x-\frac{1}{2}$.

Non-existence of asymptotes: We know $c=-\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$

If one or more values of m obtained from $\phi_n(m)=0$ are such that $\phi_n'(m)=0$; whereas $\phi_{n-1}(m)\neq 0$; we get $c=+\infty$ or $-\infty$ and this corresponds to the case of tangents going further and further away from the origin as $x\to\infty$. Hence there are no asymptote corresponding to these roots of m.

To find the asymptotes of a curve, we proceed as follows:

Step I: Find $\phi_n(m)$ by putting $x=1$ and $y=m$ in the highest (n^{th}) degree terms of the equation of the curve. Similarly, find $\phi_{n-1}(m)$ by putting $x=1$ and $y=m$ in $(n-1)^{th}$ degree terms of the eqn of the curve, and so on.

Step II: Solve the equation $\phi_n(m)=0$ and find the real values of m. Then

(i) If all the values of m are distinct, find the value of c using the equation $c\,\phi_n'(m)+\phi_{n-1}(m)=0$

(ii) If two values of m are equal then the values of c corresponding to these equation, we have equal values of m are obtained from the eqn

$$\frac{c^2}{\lfloor 2}\phi_n''(m)+c\,\phi_{n-1}'(m)+\phi_{n-2}(m)=0$$

(iii) If three values of m are equal then the values of c corresponding to these equal values of m are obtained from the equation $\frac{c^3}{\lfloor 3}\phi_n'''(m)+\frac{c^2}{\lfloor 2}\phi_{n-1}''(m)+c\,\phi_{n-2}'(m)+\phi_{n-3}(m)=0$

Remarks: (1) A curve of degree n can not have more than n asymptotes because the equation $\phi_n(m)=0$ will give at most n values of m and each value of m gives the asymptote.

(ii) Sometimes corresponding to even a real roots, there may be no asymptote. For example - $y^2=4x$

Exercise: Find the asymptote of the curve $y^3=x^2+3x$.

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Lecture - (3).

Asymptotes :-

Two parallel asymptotes :-

If any two values of m obtained from $\phi_n(m)=0$ are equal then curve is said to have two parallel asymptotes. In such cases the repeated values of m are equal, then obtained from $\phi_n(m)=0$ makes $\phi_n'(m)$ as well as $\phi_{n-1}(m)$ equal to zero.

Therefore, the equation $c\phi_n'(m)+\phi_{n-1}(m)=0$ from which the values of c is obtained, takes the form $c.0+0=0$. and does not determine any values of c.

So to determine c, when two values of m are equal we use the eqn $\frac{c^2}{L^2}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)=0$

which is obtained by equating to zero the coefficients of x^{n-2} in eqn (1) of the determination of asymptotes by shorter method.

This eqn gives two values of c corresponding to two equal values of m. Thus we get a pair of parallel asymptote.

Three or more parallel asymptote :-

The curve is said to have three parallel asymptote if the equation $\phi_n(m)=0$ has three equal roots. In this case values of c is determined from the equation

$$\frac{c^3}{\lfloor 3}\phi_n'''+\frac{c^2}{\lfloor 2}\phi_{n-1}''+c\,\phi_{n-2}'(m)+\phi_{n-3}=0$$

The method can be generalized in case of more than three parallel asymptote.

Asymptote parallel to the coordinate axes:-

Let the equation of curve of nth degree be

$$(a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\dots+a_ny^n)+(b_1x^{n-1}+\dots+b_ny^{n-1})+(c_2x^{n-2}+\dots+c_ny^{n-2})+\dots=0$$

$$\Rightarrow a_0x^n+(a_1y+b_1)x^{n-1}+(a_2y^2+b_2y+c_2)x^{n-2}+\dots=0 \quad \text{(I)}$$

As we have discussed earlier if the coefficients of x^n as well as of x^{n-1} are zero i.e. if $a_0=0$ and $a_1y+b_1=0$ will be an asymptote of the curve (I). But $a_1y+b_1=0$ will be an asymptote of the curve (I). But $a_1y+b_1=0$ is of the form $y=k$; where $k=-b_1/a_1$. Clearly,

$y=k$ is a line parallel to x-axis.

If both x^n and x^{n-1} are absent i.e. their coefficient are zero then $a_2y^2+b_2y+c_2=0$ i.e. the coefficient of $x^{n-2}=0$ will make three roots of eqn (I) infinite, and as such

$$a_2y^2+b_2y+c_2=0 \quad \text{(II)}$$

will give two asymptote parallel to the x-axis because the factor of (II) will be of the form $y=k_1$ and $y=k_2$. Thus the asymptote parallel to the x-axis may be obtained by equating to zero the coeff^ts of highest degree terms in x, provided it is not a constant.

Similarly; equate the coefficient of the highest power of y to zero, provided that it is not merely a constant.

Asymptotes of the curve that can be written in the form $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \cdots$:-

Suppose the equation of the curve is put down in the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \cdots \quad \text{--- (I)}$$

Obviously, $\frac{A}{x} + \frac{B}{x^2} + \cdots$ is convergent for sufficiently large values of x.

Differentiating (I) w.r.t. to x, we have

$$\frac{dy}{dx} = m - \frac{A}{x^2} - \frac{2B}{x^2} - \cdots$$

Hence, the equation of the tangent to (I) at (x, y) is

$$Y - y = \left(\frac{dy}{dx}\right)(X - x)$$

$$= \left(m - \frac{A}{x^2} - \frac{2B}{x^3} \cdots\right)(X - x)$$

$$Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \cdots\right)X - \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \cdots\right)x + y$$

$$\Rightarrow Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \cdots\right)X + c + \frac{2A}{x} + \frac{3B}{x^3} + \cdots \quad \text{--- (II)}$$

After substituting the values of y from (I).

When $x \to \infty$, the equation (II) becomes

$$Y = mX + c$$

Hence, the asymptote of the given curve is

$$y = mx + c.$$

Example :- Find all the asymptotes of the curve

$$(x^2 - y^2)(x + 2y) = y^2 y + 1$$

Solution: The given equation can be factorised as

$$(x+y)(x-y)(x+2y) = y^2 - y + 1$$

One asymptote corresponding to the factor $(x+y)$ is

$$x+y = \lim_{\substack{x\to\infty \\ \frac{y}{x}\to -1}} \left[\frac{y^2-y+1}{(x-y)(x+2y)}\right] = -\frac{1}{2} \quad \text{—(I)}$$

The second asymptote corresponding to the factor $(x-y)$ is

$$x-y = \lim_{\substack{x\to\infty \\ \frac{y}{x}\to +1}} \left[\frac{y^2-y+1}{(x+y)(x+2y)}\right] = \frac{1}{6} \quad \text{—(II)}$$

The third asymptote corresponding to the factor $x+2y$ is

$$x+2y = \lim_{\substack{x\to\infty \\ \frac{y}{x}\to -\frac{1}{2}}} \left[\frac{y^2-y+1}{(x+y)(x-y)}\right]$$

$$= 1/3 \quad \text{—— (III)}$$

The asymptotes are

$$2x+2y+1=0, \quad 6x-6y-1=0, \quad 3x+6y-1=0.$$

Exercise: Find the asymptote of the curve

$$(x^2-y^2)(y^2-4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$$

Asymptotes

Lecture - 4

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Alternative method of finding asymptotes of algebraic curves :—

Method I :- Let the equation of the curve be of degree n.

Let $(y-ax)$ be a non-repeated factor of the nth degree terms of the equation of the curve, then the equation of the curve can be written as

$$(y-ax)L_{n-1} + M_{n-1} = 0 \quad \text{——— (1)}$$

Where L_{n-1} is of $(n-1)$th degree and M_{n-1} is of other degrees but neither of them is greater than $(n-1)$th degree. It is obvious that in this case values of m is a and hence there is an asymptote $y = ax + c$, where c is a constant to be determined.

We have

$$c = \lim_{x\to\infty}(y-mx) = \lim_{x\to\infty}(y-ax) \quad \text{——— (ii)}$$

Where (x,y) lies on (1) and so we have

$$y - ax = -\frac{M_{n-1}}{L_{n-1}} \quad \text{——— (iii) (from (1))}$$

Hence $c = \lim_{x\to\infty}(y-ax) = \lim_{x\to\infty}\left(-\frac{M_{n-1}}{L_{n-1}}\right)$ ——— (iv)

(from (ii) & (iii))

and this limit easily calculated as $\lim_{x\to\infty}(y/x) = m$.

Method II :– Let the equation of the curve be,

$$(y-ax)^2 L_{n-2} + (y-ax) M_{n-2} + N_{n-2} = 0 \quad \text{--- (I)}$$

Where L_{n-2} and M_{n-2} contains only terms of $(n-2)$ degree and N_{n-2} has terms of other degrees; but none of them is of higher degree than $(n-2)$.

Dividing (I) by L_{n-2}, we have

$$(y-ax)^2 + (y-ax)\left\{\frac{M_{n-2}}{L_{n-2}}\right\} + \left\{\frac{N_{n-2}}{L_{n-2}}\right\} = 0$$

Taking limit when $x \to \infty$ and $\frac{y}{x} \to a$; we have

$$(y-ax)^2 + S(y-ax) + T = 0 \quad \text{--- (II)}$$

Where S and T are constants. Solving (II) for $(y-ax)$, we shall obtain two asymptotes

$$y-ax = c_1 \text{ and } y-ax = c_2$$

The above method can be applied if the nth degree terms contains $(y-ax)^2$ or a higher power of $(y-ax)$ as a factor.

Method III :– Let the equation of the curve be,

$$(ax+by+c) L_{n-1} + M_{n-1} = 0 \quad \text{--- (I)}$$

Where L_{n-1} and M_{n-1} have such terms as neither of them is of a higher degree than $(n-1)$th and L_{n-1} has at least one term of $(n-1)$ degree (which only shows that the equation of the curve is of nth degree), then the asymptote corresponding to the factor $(ax+by+c)$ is

$$ax+by+c + \lim_{\substack{x\to\infty \\ \frac{y}{x}\to -\frac{a}{b}}} \left\{\frac{M_{n-1}}{L_{n-1}}\right\} = 0.$$

Example :- (1) Find all the asymptotes of the curve
$$x^2(x+y)(x-y)^2 + ax^2(x-y) - a^2y^3 = 0$$

Solution :- Given curve is
$$x^2(x+y)(x-y)^2 + ax^2(x-y) - a^2y^3 = 0.$$

Dividing each terms of the eqn by $x^2(x+y)$; then the asymptotes corresponding to the factor $(x-y)^2$ are given by

$$(x-y)^2 + (x-y)\lim_{\substack{x\to\infty \\ \frac{y}{x}\to 1}}\left[\frac{a}{x+y}\right] - \lim_{\substack{x\to\infty \\ \frac{y}{x}\to 1}}\left[\frac{a^2y^3}{x^2(x+y)}\right] = 0$$

$$\Rightarrow (x-y)^2 + (x-y)\lim_{\substack{x\to\infty \\ \frac{y}{x}\to 1}}\left[\frac{ax}{1+(y/x)}\right] - \lim_{\substack{x\to\infty \\ \frac{y}{x}\to 1}}\left[\frac{a^2(y/x)^3}{1+y/x}\right] = 0$$

$$\Rightarrow (x-y)^2 + (x-y)\cdot(0) - \frac{a^2}{2} = 0$$

$$\Rightarrow x-y = \pm\frac{a}{\sqrt{2}}$$

The asymptotes corresponding to the factor $(x+y)$ is given by
$$x+y = \lim_{\substack{x\to\infty \\ \frac{y}{x}\to -1}}\left[\frac{-ax^2(x-y)+a^2y^3}{x^2(x-y)^2}\right]$$

$$\Rightarrow x+y = \lim_{\substack{x\to\infty \\ \frac{y}{x}\to -1}}\left[\frac{-a(1-y/x)\left(\frac{1}{x}\right)+a^2\left(\frac{y}{x}\right)^3\cdot\left(\frac{1}{x}\right)}{[1-(y/x)]^2}\right]$$

$$= 0 \Rightarrow x+y = 0$$

There is no asymptote corresponding to the factor x^2

, since the coeffts of highest power of x is constant.
Hence the required asymptotes are

$$x-y=\pm a\sqrt{2}\ ;\quad x+y=0.$$

Example ② :- Find the asymptotes of the curve.

$$(x+y)(x^4+y^4)-a(x^4+a^4)=0$$

Solution :- The given equation is of fifth degree and it may be written as

$$(x+y)L_{5-1}+M_{5-1}=0$$

Hence the asymptote is parallel to $x+y=0$

The eqn of asymptote is

$$x+y=\lim_{\substack{x\to\infty\\ \frac{y}{x}\to-1}}\left[\frac{a\{1+(a/x)^4\}}{1+(y/x)^4}\right]$$

$$=\frac{a}{2}$$

$$2x+2y=a.$$

From the equation of the curve.

$\phi_5(m)=(1+m)(1+m^4)$ (by putting $x=1$; $y=m$ in the highest degree terms). Now $\phi_5(m)=0$ gives $m=-1$ and the other four values of m are imaginary.
Hence the only real asymptote of the curve is

$$2x+2y=a.$$

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Asymptotes
Lecture-5 :-

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Intersection of the curve and its asymptotes :-

Let there be a curve of degree n and let $ax+by+c=0$ —— (I) be an asymptote of the curve, then the equation of the curve can be expressed in the form

$$(ax+by+c)L_{n-1}+L_{n-2}=0 \quad \text{—— (II)}$$

where L_{n-1} has terms of degree $(n-1)$ and L_{n-2} has terms of different degrees and of this no terms is of higher degrees than $(n-2)$.

The asymptotes (I) cuts (II),

$$\text{where } L_{n-2}=0 \quad \text{—— (III)}$$

putting the value zero of $ax+by+c$ from (I) & (II).

Now, the points of intersection of (I) and (II) will be the same as those of (I) & (III). The points of intersection of (I) & (III) will be $(n-2)$ in numbers because (I) is of degree one and (III) is of degree $(n-2)$.

So, one asymptote of a curve of nth degree which is not parallel to any other asymptote of the curve cuts the curve in $(n-2)$ points.

Suppose the ~~system~~ asymptotes of the curve of degree n are

$$L_n \equiv (y-m_1x-c_1)(y-m_2x-c_2)\cdots(y-m_nx-c_n)=0$$

Then the equation of that curve which has $L_n=0$ as its asymptotes can be put down as

$$L_n+M_{n-2}=0 \quad \text{—— (IV)}$$

P.T.O.

where M_{n-2} has no such term as is of a higher degree than $(n-2)$.

We know that if $S=0$ and $S'=0$ represents two curves, then $S+\lambda S'=0$ will present for every value of λ some curves that passes through the point of intersection of $S=0$ and $S'=0$.

Suppose $\lambda=-1$ for two curves $L_n+L_{n-2}=0$ and $L_n=0$ then $L_n+L_{n-2}-L_n=0$ i.e. $L_{n-2}=0$ is some curve that passes through the point of intersection of $L_n+L_{n-2}=0$ and $L_n=0$.

As L_{n-2} is of a degree not higher than $(n-2)$, therefore a curve of degree $(n-2)$, or lesser, can be made to pass through $n(n-2)$ points of intersection of a curve of degree n and its asymptotes.

Example:— Show that the asymptotes of the cubic

$$x^3-2y^3+xy(2x-y)+y(x-y)+1=0 \quad (1)$$

cut the curve again in three points which lie on the straight lines $x-y+1=0$

Solution:— Putting $x=1$ and $y=m$ in the highest (third) degree terms of the given equation of the curve we have

$$\phi_3(m)=1-2m^3+2m-m^3=0$$

$$\Rightarrow m=1,-1,-\tfrac{1}{2}$$

Also $\phi_2(m)=m-m^2$

& $c=-\dfrac{\phi_2(m)}{\phi_3'(m)}=-\dfrac{(m-m^2)}{(2-2m-6m^2)}$

Therefore, for $m = 1, -1, -\frac{1}{2}$; we get $c = 0, -1,$ and $\frac{1}{2}$ respectively.

The asymptote are

$x - y = 0$, $x + y + 1 = 0$ and $x + 2y - 1 = 0$.

The combined eqn of asymptote is

$$(x-y)(x+y+1)(x+2y-1) = 0$$

$$\Rightarrow x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 - x + y = 0 \quad \text{(11)}$$

Subtracting (11) from (1); we find that the point of intersection of the curve and its asymptotes lie on $x - y + 1 = 0$.

The curve is intersected by its asymptotes at $n(n-2)$ i.e. $3(3-2) \Rightarrow 3$ points.

Exercise: Find the eqn of straight line on which lie the three points of intersection of the curve $(x+a)y^2 = (y+b)x^2$ and its asymptotes.

Working Rule for asymptotes of polar curves:

Place the curve in the form $\frac{1}{r} = f(\theta)$; Put $f(\theta) = 0$ and find the values of θ from this equation. Suppose, α is one of these values.

Find $f'(\theta)$ and then obtain $f'(\alpha)$ by putting α for θ. Now the equation for asymptote is

$$r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}.$$

Example:- Find the asymptotes of the curve $r(1-\cos\theta)=a$.

Solution:- The given eqn of the curve can be written as

$$\frac{1}{r}=\frac{1-\cos\theta}{a}$$

$$\Rightarrow f(\theta)=\frac{1}{r}=\frac{1-\cos\theta}{a}$$

$$\text{so } f(\theta)=0 \Rightarrow \frac{1-\cos\theta}{a}=0 \Rightarrow \cos\theta=1$$

$$\Rightarrow \theta=2n\pi$$

$$\text{Also } f'(\theta)=\frac{1}{a}\sin\theta$$

$$\text{so } f'(2n\pi)=\frac{1}{a}\sin 2n\pi=0$$

It means $\frac{1}{f'(\alpha)}=\infty$; which does not tend to a finite limit.

Hence the curve has no asymptote.

Circular Asymptotes:- Let the eqn to a curve be $r=f(\theta)$ and let $\lim\limits_{\theta\to\infty} f(\theta)=a$, then the circle $r=a$ is called circular asymptote of the curve $r=f(\theta)$.

Example:- Findout the circular asymptote of the curve $r=\frac{3\theta}{\theta+1}$.

Solution:- The eqn of the curve is $r=\frac{3\theta}{\theta+1}=f(\theta)$ (say)

$$\lim_{\theta\to\infty} f(\theta)=\lim_{\theta\to\infty}\frac{3\theta}{\theta+1}=\lim_{\theta\to\infty}\frac{3}{1+\frac{1}{\theta}}=3$$

Hence $r=3$ is the circular asymptote.

Exercise:- (i) Find the asymptotes of the curve $r\theta=a$.

(ii) Find circular asymptotes of the curve

(a) $r(\theta^3+1)=\theta^3-1$ (b) $r(\theta-1)=a\theta$.